AUTOMORPHISM GROUPS OF POSITIVE ENTROPY ON PROJECTIVE THREEFOLDS

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Abstract. We prove two results about the natural representation of a group G of automorphisms of a normal projective threefold X on its second cohomology. We show that if X is minimal then G, modulo a normal subgroup of null entropy, is embedded as a Zariski-dense subset in a semi-simple real linear algebraic group of real rank ≤ 2. Next, we show that X is a complex torus if the image of G is an almost abelian group of positive rank and the kernel is infinite, unless X is equivariantly non-trivially fibred.

1. Introduction

Let X be a compact Kähler manifold. For an automorphism $g \in \mathrm{Aut}(X)$, its (topological) *entropy* $h(g) = \log \rho(g)$ is defined as the logarithm of the *spectral radius* $\rho(g)$ of the pullback action g^* on the total cohomology group of X, i.e.,

$$\rho(g) := \max\{|\lambda| \,;\, \lambda \text{ is an eigenvalue of } g^* | \oplus_{i\ge 0} H^i(X, \mathbb{C})\}.$$

By the fundamental result of Gromov and Yomdin, the above definition is equivalent to the original dynamical definition of entropy (cf. [11], [21]).

An element $g \in \mathrm{Aut}(X)$ is of *null entropy* if its (topological) entropy $h(g)$ equals 0. For a subgroup G of $\mathrm{Aut}(X)$, we define the *null subset* of G as

$$N(G) := \{g \in G \,|\, g \text{ is of null entropy, i.e., } h(g) = 0\}$$

which may *not* be a subgroup. A group $G \le \mathrm{Aut}(X)$ is of *null entropy* if every $g \in G$ is of null entropy, i.e., if G equals $N(G)$.

By the classification of surfaces, a complex surface S has some $g \in \mathrm{Aut}(S)$ of positive entropy only if S is bimeromorphic to a rational surface, complex torus, $K3$ surface or Enriques surface (cf. [5]). See [24] for a similar phenomenon in higher dimensions.

Recall that a normal projective variety X is *minimal* if it has at worst terminal singularities and the canonical divisor K_X is nef (cf. [12, Definition 2.34]). Let $\mathrm{NS}(X)$ be the *Neron-Severi group* and $\mathrm{NS}_{\mathbb{C}}(X) := \mathrm{NS}(X) \otimes \mathbb{C}$. For a subgroup G of $\mathrm{Aut}(X)$, let $\overline{G} \subseteq \mathrm{GL}(\mathrm{NS}_{\mathbb{C}}(X))$ be the Zariski-closure of the action of G on $\mathrm{NS}_{\mathbb{C}}(X)$ (simply denoted by $G \,|\, \mathrm{NS}_{\mathbb{C}}(X)$), and let $R(\overline{G})$ be its solvable radical, both of which are defined

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over $\mathbb{Q}$ (cf. [14, ChI; 0.11, 0.23]). We have a natural composition of homomorphisms: $\iota : G \to G \mid \mathrm{NS}_{\mathbb{C}}(X) \to \overline{G}$. Denote by

$$R(G) := \iota^{-1}(\iota(G) \cap R(\overline{G})) \lhd G.$$

Theorem 1.1. *Let X be a 3-dimensional minimal projective variety and $G \le \mathrm{Aut}(X)$ a subgroup such that $G \mid \mathrm{NS}_{\mathbb{C}}(X)$ is not virtually solvable. Then $R(G) \mid \mathrm{NS}_{\mathbb{C}}(X)$ is virtually unipotent. Replacing G by a suitable finite-index subgroup, $G/R(G)$ is embedded as a Zariski-dense subgroup in $H := \overline{G}/R(\overline{G})$ so that $H(\mathbb{R})$ is a semi-simple real linear algebraic group and is either of real rank* 1 (cf. [14, 0.25]) *or locally isomorphic to* $\mathrm{SL}_3(\mathbb{R})$ *or* $\mathrm{SL}_3(\mathbb{C})$ (*where locally isomorphic means: having isomorphic Lie algebras*).

The key step of the proof is Theorem 4.2 of which part (1) is a consequence of [7, Theorem 5.1] in which the authors have determined the actions of irreducible lattices in semi-simple real Lie groups of higher rank on threefolds.

For a subgroup G of $\mathrm{Aut}(X)$, the pair (X, G) is *non-strongly-primitive*, if there are X' bimeromorphic to X, a finite-index subgroup G_1 of G and a holomorphic map $X' \to Y$ with $0 < \dim Y < \dim X$, such that the induced bimeromorphic action of G_1 on X' is biholomorphic and descends to an action on Y with $X' \to Y$ being G_1-equivariant. (X, G) is *strongly primitive* if it is not non-strongly-primitive.

Our second main result is Theorem 1.2 (being generalized to higher dimensions in [9]).

Theorem 1.2. *Let X be a 3-dimensional normal projective variety with only $\mathbb{Q}$-factorial terminal singularities, and $G \le \mathrm{Aut}(X)$ a subgroup such that $G_0 := G \cap \mathrm{Aut}_0(X)$ is infinite and the quotient group G/G_0 is an almost abelian group of positive rank* (*cf.* 2.1 *for the terminology*). *Suppose that the pair (X, G) is strongly primitive. Then X is a complex* 3*-torus and G_0 is Zariski-dense in* $\mathrm{Aut}_0(X)$.

We remark that the almost abelian condition (as defined in 2.1) on G/G_0 is used to show that the extremal rays on X are G-periodic (cf. [25, Theorem 2.13, or Appendix]).

Corollary 1.3. *Let X be a 3-dimensional normal projective variety with only $\mathbb{Q}$-factorial rational singularities, and $G \le \mathrm{Aut}(X)$ a subgroup of* null entropy *such that $G \mid \mathrm{NS}_{\mathbb{C}}(X)$ is almost abelian of positive rank. Assume that (X, G) is strongly primitive. Then,* $\mathrm{Aut}_0(X) = \{\mathrm{id}_X\}$ *and* $h^1(X, \mathcal{O}_X) = 0$.

We do not have any example satisfying all the hypotheses of Corollary 1.3.

Let τ be a primitive cubic or quartic root of 1, $E := \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ and $X := E^n/\langle \mathrm{diag}[\tau, \dots, \tau] \rangle$ (cf. [17, Thm (0.3)], [26, Ex 1.7]). Take some g in $\mathrm{SL}_n(\mathbb{Z})$ such that g acts on X as an

automorphism of infinite order and null entropy. When $n = 3$, the group $G := \langle g \rangle$ satisfies the hypotheses of Corollary 1.3, except the strong primitivity which seems hard to verify. The action of $\langle g \rangle$ on E^n is not strongly primitive (cf. Proof of Cor. 3.8).

Remark 1.4. (1) In Theorem 1.1, suppose that $c_1(X) \ne 0$. Then the Iitaka fibration $X \to Y$ is G-equivariant, and also non-trivial by the abundance theorem or the classification of surfaces. Replacing G by a subgroup of finite index, we may assume that the induced action of G on Y is trivial (cf. [20, Theorem 14.10]), so G acts faithfully on a general fibre S and the group $G \,|\, S$ is not of null entropy (since the same holds for $G \,|\, X$; see Theorem 2.2). Since $K_S = K_X|S \sim_{\mathbb{Q}} 0$, our S is a complex 2-torus, $K3$ or Enriques surface. So there is a homomorphism $G \,|\, X = G \,|\, S \to \mathrm{SO}(1, \rho(S) - 1) \le \mathrm{SL}(\mathrm{NS}_{\mathbb{R}}(S))$ (G being replaced by a subgroup of index ≤ 2) with kernel virtually contained in $\mathrm{Aut}_0(S)$ and the Picard number $\rho(S) \le 20$.

(2) In Theorem 1.2, the strong primitivity assumption on (X, G) is necessary by considering $X = S \times T$ and $G = \langle g \rangle \times \mathrm{Aut}_0(T)$ where g is of positive entropy on a $K3$ surface S, and T a homogeneous curve ($\mathbb{P}^1$ or elliptic).

(3) The projectivity of X in Theorem 1.1 is used in applying the characterization of a quotient of an abelian threefold (cf. [18]). The projectivity of X in Theorem 1.2 is used in running the minimal model program (only for uniruled varieties).

The following Theorem 1.5 is a direct consequence of [23, Theorem 1.1]; see also the discussion in [6, §6]. It extends the classical Tits alternative [19, Theorem 1].

A compact Kähler manifold X is *ruled* if it is bimeromorphic to a manifold with a $\mathbb{P}^1$-fibration. By a result of Matsumura, X is ruled if $\mathrm{Aut}_0(X)$ is not a compact torus (cf. [10, Proposition 5.10]). When X is a compact complex Kähler manifold (or a normal projective variety), set $L := H^2(X, \mathbb{Z})/(\text{torsion})$ (resp. $L := \mathrm{NS}(X)/(\text{torsion})$), $L_{\mathbb{R}} := L \otimes_{\mathbb{Z}} \mathbb{R}$, and $L_{\mathbb{C}} := L \otimes_{\mathbb{Z}} \mathbb{C}$.

Theorem 1.5. *Let X be a compact Kähler* (*resp. projective*) *manifold of dimension n and $G \le \mathrm{Aut}(X)$ a subgroup. Then one of the following properties holds.*

(1) *$G|L_{\mathbb{C}} \ge \mathbb{Z} * \mathbb{Z}$* (*the non-abelian free group of rank two*)*, and hence $G \ge \mathbb{Z} * \mathbb{Z}$.*

(2) *$G|L_{\mathbb{C}}$ is virtually solvable and $G \ge K \cap L(\mathrm{Aut}_0(X)) \ge \mathbb{Z} * \mathbb{Z}$ where $L(\mathrm{Aut}_0(X))$ is the linear part of* $\mathrm{Aut}_0(X)$ (cf. [10, Definition 3.1, p. 240]) *and $K = \mathrm{Ker}(G \to \mathrm{GL}(L_{\mathbb{C}}))$, so X is ruled* (cf. [10, Proposition 5.10]).

(3) *There is a finite-index solvable subgroup G_1 of G such that the null subset $N(G_1)$ of G_1 is a normal subgroup of G_1 and $G_1/N(G_1) \cong \mathbb{Z}^{\oplus r}$ for some $r \le n - 1$.*

In particular, either $G \ge \mathbb{Z} * \mathbb{Z}$ or G is virtually solvable. In Cases (2) *and* (3) *above, $G|L_{\mathbb{C}}$ is finitely generated.*

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2. Entropy and Algebraic group action

In this section, we shall recall some definitions and technical results needed in the proofs and establish some easy consequences or already known facts.

2.1. Terminology and notation are as in [12]. Below are some more conventions.

Let X be a compact complex Kähler manifold (resp. a normal projective variety). As in the introduction, set $L := H^2(X, \mathbb{Z})/(\text{torsion})$ (resp. $L := \mathrm{NS}(X)/(\text{torsion})$), $L_{\mathbb{R}} := L \otimes_{\mathbb{Z}} \mathbb{R}$, and $L_{\mathbb{C}} := L \otimes_{\mathbb{Z}} \mathbb{C}$. Let $\overline{P(X)}$ be the closure of the Kähler cone (resp. the *nef cone* $\mathrm{Nef}(X)$, i.e., the closure of the ample cone) of X. Elements in $\overline{P(X)}$ are called *nef.*

For $g \in \mathrm{Aut}(X)$, let

$$d_1(g) := \max\{|\lambda| \,;\, \lambda \text{ is an eigenvalue of } g^* \mid H^{1,1}(X)\}$$

be the *first dynamical degree* of g (cf. [8, §2.2]). By the generalization of Perron-Frobenius theorem (cf. [3]) applied to $\overline{P(X)}$, for every $g \in \mathrm{Aut}(X)$, there is a nonzero nef class L_g (not unique) such that

$$g^* L_g = d_1(g) L_g.$$

We remark that g is of null entropy if and only if so is g^{-1}; if this is the case, then for every non-trivial $(1,1)$-class M with $g^*M = \lambda M$, we have $|\lambda| = 1$.

$G \,|\, Y$ denotes a naturally (from the context) induced action of G on Y. A subvariety $Z \subset X$ is *G-periodic* if Z is stabilized by a finite-index subgroup of G. For a complex torus X (as a variety), we have $\mathrm{Aut}_{\text{variety}}(X) = T \rtimes \mathrm{Aut}_{\text{group}}(X)$ where $T = \mathrm{Aut}_0(X)$ ($\cong X$) consists of all the translations of X and $\mathrm{Aut}_{\text{group}}(X)$ is the group of bijective homomorphisms of X (as a torus).

A group G is *virtually unipotent* (resp. *virtually abelian*, or *virtually abelian of rank* r) if a finite-index subgroup G_1 of G is unipotent (resp. abelian, or isomorphic to $\mathbb{Z}^{\oplus r}$). A group G is *virtually solvable* (resp. *almost abelian*, or *almost abelian of finite rank* r, cf. definition after [16, Thm 1.2]) if it has a finite-index subgroup G_1 and an exact sequence

$$1 \to H \to G_1 \to Q \to 1$$

such that H is finite, and Q is solvable (resp. abelian, or isomorphic to $\mathbb{Z}^{\oplus r}$); by Lemma 2.4, G is almost abelian of finite rank r if and only if G is virtually abelian of rank r; replacing G_1 by a finite-index subgroup, we may assume that the conjugation action of G_1 (and hence of H) on H is trivial; so in the above definition of virtually solvable group, we may also assume $H = 1$, so that our definition here coincides with the usual definition.

Theorem 2.2 below follows from Oguiso [16, Lemma 2.5] and Tits [19, Theorem 1].

Theorem 2.2. *Let X be a compact Kähler (or projective) manifold of dimension n and G a subgroup of* $\mathrm{Aut}(X)$. *Then we have:*

(1) *Suppose that G is of null entropy. Then $G \,|\, L_{\mathbb{C}}$ is virtually unipotent and hence virtually solvable (cf.* 2.1 *for notation). Moreover, $G \,|\, L_{\mathbb{C}}$ is finitely generated.*
(2) *Suppose that $G \,|\, L_{\mathbb{C}} \geq \mathbb{Z} * \mathbb{Z}$. Then G contains an element of positive entropy.*

Let X be a compact Kähler (resp. projective) manifold of dimension n. A sequence $0 \neq L_1 \cdots L_k \in H^{k,k}(X)$ $(1 \leq k < n)$ is *quasi-nef* if it is inductively obtained in the following way: first $L_1 \in \overline{P(X)}$; once $L_1 \cdots L_{j-1} \in H^{j-1,j-1}(X)$ is defined, we define

$$L_1 \cdots L_j = \lim_{t \to \infty} L_1 \cdots L_{j-1} \cdot M_t$$

for some $M_t \in \overline{P(X)}$ (cf. [23, §2.2]). We remark that for $j \geq 2$, the L_j which appears in the construction of the (j,j)-class $L_1 \cdots L_j$, may not belong to $\overline{P(X)}$. A group $G \leq \mathrm{Aut}(X)$ is *polarized* by the quasi-nef sequence $L_1 \cdots L_k$ $(1 \leq k < n)$ if

$$g^*(L_1 \cdots L_k) = \chi_1(g) \cdots \chi_k(g)(L_1 \cdots L_k)$$

for some characters $\chi_j : G \to (\mathbb{R}_{>0}, \times)$.

Theorem 2.3 gives criteria of virtual solvability, with (3) proved in [23, Theorem 1.2].

Theorem 2.3. *Let X be a compact Kähler (resp. projective) manifold of dimension n and G a subgroup of* $\mathrm{Aut}(X)$. *Then we have (cf.* 2.1 *for notation of $L_{\mathbb{C}}$) :*

(1) *Suppose that $G \,|\, L_{\mathbb{C}}$ is virtually solvable and its Zariski-closure in* $\mathrm{GL}(L_{\mathbb{C}})$ *is connected. Then G is polarized by a quasi-nef sequence $L_1 \cdots L_k$ $(1 \leq k < n)$.*
(2) *Conversely, suppose that G is polarized by a quasi-nef sequence $L_1 \cdots L_k$ $(1 \leq k < n)$. Then $G \,|\, L_{\mathbb{C}}$ is virtually solvable.*
(3) *$G \,|\, L_{\mathbb{C}}$ is virtually solvable if and only if there exists a finite-index subgroup G_1 of G such that $N(G_1) \lhd G_1$ and $G_1/N(G_1) \cong \mathbb{Z}^{\oplus r}$ for some $r \leq n-1$.*

We need the following lemmas for the proof of theorems.

Lemma 2.4. *Let G be a group, and $H \lhd G$ a finite normal subgroup. Then we have:*

(1) *Suppose that for some $r \geq 1$ and $g_i \in G$ we have:*

$$G/H = \langle \bar{g}_1 \rangle \times \cdots \times \langle \bar{g}_r \rangle \cong \mathbb{Z}^{\oplus r}.$$

Then there is an integer $s > 0$ such that the subgroup $G_1 := \langle g_1^s, \ldots, g_r^s \rangle$ satisfies

$$G_1 = \langle g_1^s \rangle \times \cdots \times \langle g_r^s \rangle \cong \mathbb{Z}^{\oplus r}$$

and is of finite-index in G; further, the quotient map $\gamma : G \to G/H$ restricts to an isomorphism $\gamma \,|\, G_1 : G_1 \to \gamma(G_1)$ onto a finite-index subgroup of G/H.

(2) *A group is almost abelian of finite rank r if and only if it is virtually abelian of rank r.*

Proof. (2) follows from (1). For (1), we only need to find $s > 0$ such that g_i^s commutes with g_j^s for all i, j. Since G/H is abelian, the commutator subgroup $[G, G] \leq H$. Thus the commutators $[g_1^t, g_2]$ $(t > 0)$ all belong to H. The finiteness of H implies that $[g_1^{t_1}, g_2] = [g_1^{t_2}, g_2]$ for some $t_2 > t_1$, which implies that $g_1^{s_{12}}$ commutes with g_2, where $s_{12} := t_2 - t_1$. Similarly, we can find an integer $s_{1j} > 0$ such that $g_1^{s_{1j}}$ commutes with g_j. Set $s_1 := s_{12} \times \cdots \times s_{1r}$. Then $g_1^{s_1}$ commutes with every g_j. Similarly, for each i, we can find an integer $s_i > 0$ such that $g_i^{s_i}$ commutes with g_j for all j. Now $s := s_1 \times \cdots \times s_r$ will do the job. This proves the lemma. □

Lemma 2.5. *Let X be a compact complex Kähler manifold (resp. normal projective variety), and G a subgroup of $\mathrm{Aut}(X)$. Then, replacing G by a suitable finite-index subgroup, the following are true (cf. 2.1 for notation $L_{\mathbb{C}}$).*

(1) *There is a normal subgroup $U \lhd G$ such that $U \,|\, L_{\mathbb{C}}$ is unipotent and G/U is embedded as a Zariski-dense subgroup in a reductive complex linear algebraic group.*
(2) *There is a normal subgroup $R \lhd G$ such that $R \,|\, L_{\mathbb{C}}$ is solvable and G/R is embedded as a Zariski-dense subgroup in a semi-simple complex linear algebraic group H.*

Proof. Let $\overline{G}$ be the Zariski-closure of $G \,|\, L_{\mathbb{C}} \subseteq \mathrm{GL}(L_{\mathbb{C}})$, and ι the composite: $G \to G \,|\, L_{\mathbb{C}} \to \overline{G}$. Replacing G by the intersection of G and the ι-inverse of the identity connected component of $\overline{G}$, we may assume that $\overline{G}$ is connected. Let $U(\overline{G})$ (resp. $R(\overline{G})$) be the unipotent radical (resp. the radical) of $\overline{G}$. Let $U \leq G$ (resp. $R \leq G$) be the ι-inverse of $U(\overline{G})$ (resp. $R(\overline{G})$). Then the embeddings $G/U \to \overline{G}/U(\overline{G})$ and $G/R \to \overline{G}/R(\overline{G})$, and U and R here meet the requirements of the lemma. □

Lemma 2.6. *We use the notation $L_{\mathbb{C}}$ of 2.1. A group $G \leq \mathrm{Aut}(X)$ has finite restriction $G \,|\, L_{\mathbb{C}}$ if and only if the index $|G : G \cap \mathrm{Aut}_0(X)|$ is finite.*

Proof. Consider the exact sequence

$$1 \to K \to G \to G \,|\, L_{\mathbb{C}} \to 1.$$

For an ample divisor or Kähler class ω of X, our K is a subgroup of $\mathrm{Aut}_\omega(X) := \{g \in \mathrm{Aut}(X) \,|\, g^*\omega = \omega\}$, where the latter contains $\mathrm{Aut}_0(X)$ as a group of finite-index (cf. [13, Proposition 2.2]). Now the last group below is a finite group

$$K/(K \cap \mathrm{Aut}_0(X)) \cong (K\ \mathrm{Aut}_0(X))/\,\mathrm{Aut}_0(X) \leq \mathrm{Aut}_\omega(X)/\,\mathrm{Aut}_0(X).$$

The lemma follows since the connected group $\mathrm{Aut}_0(X)$ acts trivially on the lattice L (and hence on $L_{\mathbb{C}}$) so that $G \cap \mathrm{Aut}_0(X) = K \cap \mathrm{Aut}_0(X)$. □

A more precise version of 2.7 below was proved in [6, 6.1] for finitely generated groups.

Lemma 2.7. *Let G be a group of automorphisms of a compact Kähler manifold X. Consider an exact sequence of groups:*

$$1 \to N \to G \to Q \to 1$$

Suppose N is contained in the union of finitely many connected components of $\mathrm{Aut}(X)$. *Suppose both N and Q are virtually solvable. Then G is also virtually solvable.*

Proof. Let $\overline{N} \subseteq \mathrm{Aut}(X)$ be the Zariski-closure of N. Replacing G by a suitable finite-index subgroup, we may assume that Q is solvable. Since $\overline{N} \cap G \lhd G$, we have $(\overline{N})_0 \cap G \lhd G$ for the identity connected component $(\overline{N})_0$ of $\overline{N}$. Hence $M := (\overline{N})_0 \cap N \lhd G$. Now

$$N/M \cong (N\,(\overline{N})_0)/(\overline{N})_0 \leq \overline{N}/(\overline{N})_0$$

where the latter is a finite group. We have an exact sequence

$$1 \to N/M \to G/M \to G/N = Q \to 1.$$

Replacing G/M by a finite-index subgroup we may assume that the conjugate action of G/M (and hence of N/M) on the finite group N/M is trivial. Thus N/M is abelian and hence G/M is solvable. Since N is virtually solvable so is $\overline{N}$. Hence $(\overline{N})_0$ is solvable. Thus M is solvable. Therefore, G is solvable. □

Lemma 2.8. *Let X be a compact complex Kähler manifold* (*resp. normal projective variety*), *and $G \leq \mathrm{Aut}(X)$ a subgroup. Assume the following two conditions:*

1. *$H \lhd G$; and H has a finite-index subgroup H_1 such that the null set $N(H_1)$ is a* (*normal*) *subgroup of H_1 and $H_1/N(H_1) = \langle \overline{h} \rangle \cong \mathbb{Z}$ for some $h \in H_1$.*
2. *Suppose that there is a common nef eigenvector L_1 of H_1, and further that $h^*L_1 = d_1(h)L_1$, i.e., L_1 equals some L_h up to scalar; suppose also that for every $s \neq 0$ and every nef M so that $(h^s)^*M = \lambda M$ with $\lambda \neq 1$, we have M parallel to either one of $L_{h^{\pm 1}}$* (*which are two fixed nef eigenvectors*).

Then the stabilizer subgroup $\mathrm{Stab}_{L_h}(G) := \{g \in G \,|\, g^*L_h$ *is parallel to* $L_h\}$ *has index* ≤ 2 *in G.*

Proof. We begin with:

Claim 2.9. *For every $g \in G$, the class g^*L_h is parallel to one of $L_{h^{\pm 1}}$.*

We prove the claim. Take any element g of G. Since $H \lhd G$, we have $ghg^{-1} \in H \setminus N(H)$. Hence $gh^a g^{-1}$ is in H_1 with $a = |H : H_1|$, so it equals $h^b n$ for some $b \neq 0$ and $n \in N(H_1)$. Since the element n of $N(H_1)$ fixes $L_1 = L_h$ (cf. 2.1),

$$(gh^a g^{-1})^* L_h = d_1(h)^b L_h, \quad (h^a)^*(g^* L_h) = d_1(h)^b (g^* L_h).$$

Now the condition (2) applied to h^a and $\lambda := d_1(h)^b \neq 1$, implies the claim.

Return to the proof of Lemma 2.8. Suppose there is some $g_1 \in G \backslash \mathrm{Stab}_G(L_h)$. Take any $g \in G \backslash \mathrm{Stab}_G(L_h)$. By Claim 2.9, we have (with equalities all up to scalars) $g^* L_h = L_{h^{-1}} = g_1^* L_h$, $(g g_1^{-1})^* L_h = (g_1^{-1})^* g^* L_h = L_h$. Hence $g g_1^{-1} \in \mathrm{Stab}_G(L_h)$ and $g = (g g_1^{-1}) g_1 \in \mathrm{Stab}_G(L_h)\, g_1$. So $G = \mathrm{Stab}_G(L_h) \cup \mathrm{Stab}_G(L_h)\, g_1$. The lemma follows. □

2.10. Proof of Theorem 2.2

Assertion (2) follows from Assertion (1), so we only need to prove Theorem 2.2(1).

We follow the proof of Oguiso [16, Prop 2.2]. Since G is of null entropy, the subset

$$U := \{g \in G\,;\, g \,|\, L_{\mathbb{C}} \ \text{ is unipotent}\}$$

is a normal subgroup of G. If $G \,|\, L_{\mathbb{C}}$ is not virtually solvable, then by the classical Tits alternative theorem [19, Theorem 1], there are $g_i \in G$ such that $\langle g_1, g_2 \rangle | L_{\mathbb{C}} = (\langle g_1 \rangle | L_{\mathbb{C}}) * (\langle g_2 \rangle | L_{\mathbb{C}}) = \mathbb{Z} * \mathbb{Z}$. As observed in [16], $g_i^s \in U$ for some $s \geq 1$, and hence $\mathbb{Z} * \mathbb{Z} = \langle g_1^s, g_2^s \rangle \,|\, L_{\mathbb{C}} \leq U \,|\, L_{\mathbb{C}}$ which is unipotent (and hence solvable). This is absurd.

Thus, $G \,|\, L_{\mathbb{C}}$ is virtually solvable. Replacing G by a suitable finite-index subgroup, we may assume that $G \,|\, L_{\mathbb{C}}$ is solvable and its closure $\overline{G}$ in $\mathrm{GL}(L_{\mathbb{C}})$ is connected (and solvable). Write $\overline{G} = \overline{U} \rtimes \overline{T}$ where $\overline{U}$ is the unipotent radical and $\overline{T}$ a maximal torus in $\overline{G}$. As observed in [16], the image of G via the quotient map $\overline{G} \to \overline{T}$ is a torsion group in $\mathrm{GL}(L_{\mathbb{C}})$ with bounded exponent and hence a finite group by Burnside's theorem. Thus the index $|G : U| < \infty$.

To finish the proof of Assertion (1), we may assume that $G = U$ and it suffices to show that $G \,|\, L$ is generated by $\ell(\ell-1)/2$ elements where $\ell = \mathrm{rank}\, L$. Regarding $\overline{G}$ as a subgroup of upper triangular matrices, there is a standard normal series

$$1 \lhd U_1 \lhd U_2 \lhd \cdots \lhd U_{\ell(\ell-1)/2} = \overline{G}$$

such that the factor groups are all 1-dimensional. Restricting the series to $G \,|\, L$, we get a normal series of discrete groups whose factor groups are cyclic groups. Thus $G \,|\, L$ is generated by $\ell(\ell-1)/2$ elements. This proves Theorem 2.2.

2.11. Proof of Theorem 2.3

(1) was proved in [23, Theorem 1.2]. For (2), suppose that G is polarized by a quasi-nef sequence $L_1 \cdots L_k$ $(1 \leq k < n)$ so that $g^*(L_1 \cdots L_k) = \chi_1(g) \cdots \chi_k(g) L_1 \cdots L_k$. As in the proof of [23, Theorem 1.2], the homomorphism

$$\varphi : G \to (\mathbb{R}, +), \quad g \mapsto (\log \chi_1(g), \ldots, \log \chi_{n-1}(g))$$

has $\mathrm{Ker}(G) = N(G)$, and $\varphi(G) = \mathbb{Z}^{\oplus r}$ a lattice in $\mathbb{R}^{n-1}$. By Theorem 2.2, $N(G) \,|\, L_{\mathbb{C}}$ is virtually solvable, so is $G \,|\, L_{\mathbb{C}}$, since $G/N(G)$ is abelian and by Lemma 2.7.

For (3), the "if part" follows from Theorem 2.2 and Lemma 2.7. The "only if" part is by [23, Theorem 1.2, Remark 1.3]. This proves Theorem 2.3.

2.12. Proof of Theorem 1.5

We may assume that Assertion (1) is not satisfied. Replacing G by a suitable finite-index subgroup and by [19, Thm 1], we may assume that $G|L_{\mathbb{C}}$ is solvable and its closure $\overline{G}$ in $\mathrm{GL}(L_{\mathbb{C}})$ is connected (and solvable). Let $K = \mathrm{Ker}(G \to G|L_{\mathbb{C}})$ be as in Lemma 2.6.

Suppose that K is virtually solvable. Then so is G by Lemma 2.7. Thus Theorem 1.5(3) occurs, by [23, Theorem 1.2, Remark 1.3].

Suppose that K is not virtually solvable. Consider the exact sequence

$$1 \to L_A \to \mathrm{Aut}_0(X) \to T \to 1$$

where L_A is the linear part of $\mathrm{Aut}_0(X)$ and T a compact complex torus (cf. [13, Theorem 3.12]). This induces the exact sequence (with Q abelian):

$$1 \to K \cap L_A \to K \cap \mathrm{Aut}_0(X) \to Q \to 1.$$

We may assume that Theorem 1.5(2) does not occur. So $K \cap L_A$ is virtually solvable by Tits alternative. Thus so is $K \cap \mathrm{Aut}_0(X)$ by the exact sequence above and Lemma 2.7. Now $K/(K \cap \mathrm{Aut}_0(X))$ is a finite group by Lemma 2.6. Hence K is also virtually solvable, contradicting our extra assumption.

For the final assertion, replacing G by a suitable finite-index subgroup, we may assume that $G \,|\, L_{\mathbb{C}}$ is solvable and has connected Zariski-closure in $\mathrm{GL}(L_{\mathbb{C}})$. Then $G/N(G) \cong \mathbb{Z}^{\oplus r}$ by [23, Theorem 1.2]. This and Theorem 2.2 for $N(G)$ imply the assertion.

3. Strong primitivity for threefolds

We prove Theorem 1.2. Replacing G_0 by the identity connected component of its Zariski-closure in $\mathrm{Aut}_0(X)$ we may further assume that $G_0 = G \cap \mathrm{Aut}_0(X)$ is connected, positive-dimensional and closed in $\mathrm{Aut}_0(X)$. Because G acts naturally on the quotient of X modulo G_0 and because of our assumption, we may assume that one orbit of G_0 is a Zariski-dense open subset of X, i.e., X is almost homogeneous (cf. [23, Lemma 2.14]).

Claim 3.1. *Suppose the irregularity $q(X) = h^1(X, \mathcal{O}_X) > 0$. Then Theorem* 1.2 *is true.*

We prove Claim 3.1. By the proof of [23, Lemma 2.13], the albanese map $a : X \to A := \mathrm{Alb}(X)$ is surjective, birational and necessarily $\mathrm{Aut}(X)$-equivariant. Our G_0 induces an action on A and we denote it by $G_0|A$. Since $G_0|A$ also has a Zariski-dense open orbit in A, we have $G_0|A = \mathrm{Aut}_0(X)$ ($\cong A$). Let $B \subset A$ be the locus over which a is not an isomorphism. Note that B and $a^{-1}(B)$ are G_0-stable. Since $G_0|A = \mathrm{Aut}_0(X)$, we have $B = \emptyset$. Claim 3.1 is proved.

We continue the proof of Theorem 1.2. By Claim 3.1, we may assume that $q(X) = 0$. Thus $G_0 \le \mathrm{Aut}_0(X)$ is a linear algebraic group and has a Zariski-dense open orbit in X. In particular, X is ruled and unirational, because linear algebraic groups are rational varieties by a classical result of Chevalley.

In the rest of the proof, we shall derive a contradiction. Let $U \subseteq X$ be the open dense G_0-orbit and $F := X \setminus U$. Then F consists of finitely many prime divisors and some subvarieties of codimension ≥ 2. Since $G_0 \lhd G$, we may assume that both U and all irreducible components of F are G-stable, after replacing G by a suitable finite-index subgroup. X has only finitely many G_0-periodic prime divisors, all of which are contained in F and G-stable.

By the minimal model program (MMP) in dimension three (cf. [12, §3.31, §3.46]), the end product of a uniruled variety (like our X here) is an extremal Fano contraction $f : X_m \to Y$ with a general fibre $X_{m,y}$, i.e., by definition, the restriction $-K_{X_m} \mid X_{m,y}$ of the canonical divisor $-K_{X_m}$ is ample and the Picard numbers satisfy $\rho(X_m) = 1 + \rho(Y)$.

Claim 3.2. (1) *Every G_0-periodic subvariety of X is actually G_0-stable.*

(2) *There are a composite $X = X_0 \dashrightarrow X_1 \cdots \dashrightarrow X_m$ of birational extremal contractions and an extremal Fano contraction $X_m \to Y$ with* $\dim Y < \dim X$. *The induced birational action of G_0 on each X_i is biregular. $G_0|X_m$ descends to an action on Y so that $X_m \to Y$ is G_0-equivariant.*

(3) *In (2), for every finite-index subgroup G_1 of G, there is at least one $i \in \{1, \dots, m\}$ such that the induced action of G_1 on X_i is not biregular.*

(4) *In (2), let $s \le m$ be the largest integer such that $X_i \to X_{i+1}$ is divisorial for every $i \in \{0, 1, \dots, s-1\}$. Then, replacing G by a suitable finite-index subgroup, the induced birational action of G on each X_i $(i < s)$ is biregular and hence each map $X_{i-1} \to X_i$ is G-equivariant. In particular, $s < m$.*

Remark 3.3. By the choice of s in (4), $X_s \dashrightarrow X_{s+1}$ is a flip with a flipping contraction $X_s \to Y_s$ and with $X_{s+1} = \mathrm{Proj}_{Y_s}(\oplus_{m \ge 0} \mathcal{O}_{Y_s}(mK_{Y_s}))$ (cf. [12, Cor 6.4 or Thm 3.52]).

We now prove Claim 3.2. (1) is true because G_0 is a connected group. For the first part of (2), see [12, §3.31, §3.46] when $\dim X = 3$ and [2, Corollary 1.3.2] when $\dim X$ is arbitrary. The second part of (2) is true because G_0 acts trivially on $H^i(X, \mathbb{Z})$, and also on $\mathrm{NS}_{\mathbb{C}}(X)$ and the extremal rays of $\overline{\mathrm{NE}}(X)$ (cf. [25, Lemmas 2.12 and 3.6]).

For (4), suppose that $X = X_0 \to X_1$ is a divisorial contraction of an extremal ray $R := \mathbb{R}_{>0}[\ell]$ with an exceptional divisor D_0. Since G_0 acts trivially on the extremal rays of $\overline{\mathrm{NE}}(X)$, this D_0 is G_0-stable. So D_0 is contained in F and is G-stable.

Since the natural map $G/G_0 \to \mathrm{Aut}(H^2(X, \mathbb{Z}))$ has finite kernel (cf. [13, Proposition 2.2]) and by the assumption, G/G_0 is almost abelian of finite rank $r > 0$. By Lemma 2.4

and replacing G by a finite-index subgroup, there is some $H_0 \lhd G$ such that H_0 contains G_0 as a subgroup of finite index and $G/H_0 = \langle \bar{g}_1 \rangle \oplus \cdots \oplus \langle \bar{g}_r \rangle \cong \mathbb{Z}^{\oplus r}$ for some $g_i \in G$.

In [25, Lemma 3.7], it is proved that a positive power of g_i preserves the extremal ray R and hence descends to a biregular automorphism of X_1. Thus $X \to X_1$ is G-equivariant after G is replaced by a finite-index subgroup. Indeed, we may assume that $g_i(R) = R$ so that $\{g(R) \mid g \in G\}$ consists of no more than $|H_0 : G_0|$ extremal rays so that a finite-index subgroup of G fixes R. *This and the second sentence of the next paragraph are the places where we need G/G_0 to be almost abelian.*

For (3), suppose the contrary that G (replaced by a finite-index subgroup) acts biregularly on all X_i. Then, as in the proof of (4) above, by [25, Theorem 2.13, or Appendix], we may assume that G (replaced by its finite-index subgroup) fixes the extremal ray giving rise to the extremal Fano contraction $X_m \to Y$, and hence $X_m \to Y$ is G-equivariant. By the strong primitivity assumption, we have $\dim Y = 0$, so the Picard number $\rho(X_m) = 1$ and $-K_{X_m}$ is ample. Since G fixes the ample class of $-K_{X_m}$, it is a finite extension of G_0 (cf. [13, Proposition 2.2]). This contradicts the assumption. Claim 3.2 is proved.

Claim 3.4. *It is impossible that* $\mathrm{NS}_{\mathbb{C}}(X_i)$ *with* $0 \le i \le m$ *is spanned by* $-K_{X_i}$ *and* G_0*-periodic divisors, or that* $\mathrm{NS}_{\mathbb{C}}(Y)$ *is spanned by* G_0*-periodic divisors.*

Indeed, note that $\mathrm{NS}_{\mathbb{C}}(X_m)$ is spanned by $-K_{X_m}$ (which is ample over Y) and the pullback of $\mathrm{NS}_{\mathbb{C}}(Y)$, and $\mathrm{NS}_{\mathbb{C}}(X)$ is spanned by the pullback of $\mathrm{NS}_{\mathbb{C}}(X_i)$ and (necessarily G_0-stable) exceptional divisors of $X \dashrightarrow X_i$. Thus we only need to rule out the possibility that $\mathrm{NS}_{\mathbb{C}}(X)$ is spanned by $-K_X$, and G_0-stable divisors D_i all of which are necessarily contained in F and hence G-stable.

Write an ample divisor M on X as a combination of $-K_X$ and D_i's. Then $G \le \mathrm{Aut}_{[M]}(X)$, so $|G/G_0| < \infty$ as in the proof of Lemma 2.6, contradicting the assumption.

Claim 3.5. X_m *and hence* Y *contain a* G_0*-fixed point* (*here we use that* $\dim X = 3$).

Indeed, note that a smooth threefold has no flip and a flip preserves the singularity type of a threefold. By Claim 3.2, for some $m - 1 \ge t \ge s$, $X_t \dashrightarrow X_{t+1}$ is a flip and $X_{t+1} \to \cdots \to X_m$ is the composite of extremal divisorial contractions. So the non-empty finite set $\mathrm{Sing}\, X_{t+1}$ (cf. [12, Corollary 5.18]) and its image on X_m are fixed by G_0.

Claim 3.6. *It is impossible that* $\dim Y \le 1$.

Indeed, if $\dim Y = 0$, then $\mathrm{NS}_{\mathbb{C}}(X_m)$ is of rank one and spanned by $-K_{X_m}$ (which is ample over Y). This contradicts Claim 3.4. If $\dim Y = 1$ then the rank two space $\mathrm{NS}_{\mathbb{C}}(X_m)$ is spanned by $-K_{X_m}$ (which is ample over Y) and the fibre over a G_0-fixed point y_0 (cf. Claim 3.5). This contradicts Claim 3.4.

We continue the proof of Theorem 1.2. Take an extremal ray on X_s generated by a rational curve ℓ and let $X_s \dashrightarrow X_{s+1}$ be the flip (cf. Claim 3.2 for s). Note that G_0 stabilizes all irreducible components E_i of the exceptional locus of the flipping contraction $X_s \to Y_s$ and G (replaced by a finite-index subgroup) stabilizes all irreducible components D_{ij} of the Zariski closure of $\cup_{g\in G}\, g(E_i)$, because $G_0 \lhd G$. These D_{ij} are unions of 'small' G_0-orbits and hence are contained in the image of the algebraic subset $F \subset X$.

If $\dim D_{ij} = \dim E_i = 1$, then G preserves the extremal ray $\mathbb{R}_{\ge 0}[\ell] \subseteq \overline{\mathrm{NE}}(X_s)$ and we can descend G to a biregular action on X_{s+1} (cf. [25, Lemma 3.6]). Now apply MMP on X_{s+1} and continue the process.

Assume that $\dim D_{ij} = 2 > \dim E_i = 1$. If G_0 acts trivially on some g_0E_i in the set $\{gE_i \,|\, g \in G\}$, then $G_0 = gG_0g^{-1}$ acts trivially on gg_0E_i, i.e., on all $g'E_i$ ($g' \in G$). Hence $G_0 \,|\, D_{ij} = \mathrm{id}$. This contradicts Claim 3.7 below.

Suppose that G_0 acts non-trivially on some g_0E_i and hence on all gE_i ($g \in G$). Then these extremal curves gE_i are fibres of the quotient map $D_{ij} \to D_{ij}/G_0 =: B$ over a curve B, hence homologous to each other. So they give rise to one and the same class in the extremal ray $\mathbb{R}_{\ge 0}[\ell] \subseteq \overline{\mathrm{NE}}(X_s)$. Thus G preserves this extremal ray and we can descend G to a biregular action on X_{s+1} (cf. [25, Lemma 3.6]). Now apply MMP on X_{s+1} and continue the process. Therefore, we can continue the G-equivariant MMP and reach an extremal Fano fibration $X_m \to Y$ which is a contradiction (cf. Claim 3.2).

To complete the proof of Theorem 1.2, we still need to prove:

Claim 3.7. *It is impossible that* $\dim Y = 2$, $\dim D_{ij} = 2$ *and* $G_0 \,|\, D_{ij} = \mathrm{id}$.

We now prove Claim 3.7. $X_m \to Y$ is known as an extremal conic fibration. We may take a G_0-equivariant blowup $\hat{X} \to X_s$ to resolve indeterminacy of the composite $\pi_s : X_s \dashrightarrow X_m \to Y$ so that the induced map $\hat{\pi} : \hat{X} \to Y$ is holomorphic and G_0-equivariant. By [15, Theorem 4.8], there exist blowups $\sigma_x : X' \to \hat{X}$ and $\sigma_y : Y' \to Y$ with X' and Y' smooth, and extremal conic fibration $\pi' : X' \to Y'$ such that $\hat{\pi} \circ \sigma_x = \sigma_y \circ \pi'$. We may also assume that the four maps above are G_0-equivariant by taking extra blowups so that they are equivariant (noting that G_0 stabilizes extremal rays).

We will reach a contradiction to Claim 3.4. To do so, we consider both Y and Y'.

Indeed, if $K_{Y'}^2 \le 7$, then $\mathrm{NS}_{\mathbb{C}}(Y')$ (and hence $\mathrm{NS}_{\mathbb{C}}(Y)$) are spanned by G_0-stable curves (i.e., the negative curves on Y'). This contradicts Claim 3.4. Therefore, we may assume that $K_{Y'}^2 = 9$ or 8, and $Y' = \mathbb{P}^2$ or a Hirzebruch surface F_d of degree $d \ge 0$.

If $Y' = \mathbb{P}^2$ or $Y' = \mathbb{P}^1 \times \mathbb{P}^1$, then Y' has no negative curve to contract, so $Y' = Y$.

If $Y = F_d$ then G_0 stabilizes a fibre passing through a fixed point of y_0 of $G_0|Y$ (cf. Claim 3.5), and the zero-section through y_0 (resp. the unique $(-d)$-curve) when $d = 0$ (resp. $d \ge 1$). This contradicts Claim 3.4.

Therefore, we may assume that either $Y = Y' = \mathbb{P}^2$, or $F_d = Y' \to Y$ (with $d \geq 1$) is the contraction of the unique $(-d)$-curve. Thus the Picard number $\rho(X_m) = 1 + \rho(Y) = 2$.

Let $D'_{ij} \subset X'$ be the proper transform of $D_{ij} \subset X_s$. Then G_0 acts trivially on D'_{ij} because so does G_0 on D_{ij}. Since every fibre of $\pi' : X' \to Y'$ is 1-dimensional, the image $C_{ij} \subseteq Y'$ of D'_{ij} is the whole Y' or a curve, and $G_0 \,|\, C_{ij} = \mathrm{id}$. Since $G_0 \,|\, Y = \mathrm{id}$ would contradict Claim 3.4, we may assume that C_{ij} is a curve in Y'. If $Y' = Y = \mathbb{P}^2$ (resp. $Y' = F_d \to Y$ is the contraction of the $(-d)$-curve), then $G_0|Y$ stabilizes C_{ij} (resp. the image of C_{ij} or every generating line). This contradicts Claim 3.4. Claim 3.7 is proved.

Corollary 3.8. *Let X be a 3-dimensional normal projective variety and $G \leq \mathrm{Aut}(X)$ a subgroup of* null entropy *such that $G_0 := G \cap \mathrm{Aut}_0(X)$ is infinite and the quotient G/G_0 is an almost abelian group of positive rank. Then (X, G) is not strongly primitive.*

Proof. Taking a G-equivariant resolution, we may assume that X is smooth. With our assumption and the proof of Theorem 1.2, we may assume that X is a complex torus, and G_0 is connected, is closed and has a Zariski-dense open orbit in X. Thus $G_0 = \mathrm{Aut}_0(X)$. By Lemma 2.4 and replacing G by a suitable finite-index subgroup, we may assume that G/G_0 is abelian and equals $\langle \bar{g}_1, \ldots, \bar{g}_r \rangle$ for $g_i \in G$, where the order $o(\bar{g}_i) = \infty$; moreover, g_i has unipotent representation matrix on $H^0(X, \Omega^1_X)$ using Kronecker's theorem as in [24, Lemma 2.14]. Write $g_i = T_{t_i} \circ h_i$ where T_{t_i} is the translation by t_i and h_i is a group automorphism. As in [24, Lemma 2.15], the identity connected component B of the fixed locus X^{h_1} (pointwise) has dimension equal to that of $\mathrm{Ker}(h_1^* - \mathrm{id}) \subset H^0(X, \Omega^1_X)$ and is hence between 1 and $\dim X - 1$. Note that $h_1 h_j = h_j h_1$ holds modulo $\mathrm{Aut}_0(X)$ and hence holds in $\mathrm{Aut}(X)$ since both sides fix the origin. Thus $h_j(B)$ is contained in X^{h_1} and hence equals B since it contains the origin. Now $g_j(x + B) = g_j(x) + B$. So g_j permutes cosets of the quotient torus X/B; the same is true for elements of $\mathrm{Aut}_0(X)$. Thus, the quotient map $X \to X/B$ is G-equivariant. This proves Corollary 3.8. □

3.9. Proof of Corollary 1.3

Taking a G-equivariant resolution, we may assume that X is smooth. If $q(X) > 0$, then, by Claim 3.1, we may assume that X is a complex torus so that $\mathrm{Aut}_0(X) \neq \{\mathrm{id}_X\}$. Thus, we may always assume that $\mathrm{Aut}_0(X) \neq \{\mathrm{id}_X\}$.

Replacing G by $G.\,\mathrm{Aut}_0(X)$ we may assume that $G \geq G_0 := \mathrm{Aut}_0(X)$. According to Lemma 2.6, we have $G|\,\mathrm{NS}_{\mathbb{C}}(X) = G/K$ where $|K/G_0| < \infty$. Thus G/G_0 is also almost abelian of positive rank by assumption. Now Corollary 1.3 follows from Corollary 3.8.

4. Minimal threefolds

Below sufficient conditions for being a quotient of a torus are given.

Theorem 4.1. *Let X be a 3-dimensional minimal projective variety. Assume that one of the following two properties is satisfied:*

1. *The first Chern class $c_1(X) = 0$. The second Chern class $c_2(X)$ (as a linear form on $\mathrm{NS}_{\mathbb{C}}(X)$ as in* [18, p. 265]*) has zero intersection with a nef and big $\mathbb{R}$-divisor.*
2. *There is a subgroup $G \le \mathrm{Aut}(X)$ such that the null set $N(G)$ is a subgroup of G and $G/N(G) \cong \mathbb{Z}^{\oplus 2}$.*

Set $B := \mathrm{Aut}(X)$. Then there is a B-equivariant birational surjective morphism $X \to X'$ such that $X' \cong T/F$ for a finite group F acting freely outside a finite set of an abelian variety T of dimension three. Further, the action of B on X' lifts to an action of a group $\widetilde{B}$ on T such that $\widetilde{B}/F \cong B$.

Proof. Assume the condition (1) in Theorem 4.1. Let

$$D := \mathrm{Nef}(X) \cap c_2(X)^{\perp} = \{M \in \mathrm{Nef}(X) \mid M.c_2(X) = 0\}$$

be a closed subcone of the nef cone $\mathrm{Nef}(X)$ of X. Let

$$C := \overline{\mathrm{NE}}(X) \cap D^{\perp} = \{[\ell] \in \overline{\mathrm{NE}}(X) \mid \ell.D_0 = 0 \text{ for all } D_0 \in D\}$$

be a closed subcone of the closed cone $\overline{\mathrm{NE}}(X)$ of effective curves on X. Then $c_2(X) \in C$ by definition and using Miyaoka's pseudo-effectivity of c_2 for any minimal variety X of dimension n: $c_2(X) \cdot (H_1 \cdots H_{n-2}) \ge 0$ for all nef divisors H_i on X (cf. [18, Theorem 4.1, Proposition 1.1]).

By assumption, D contains a nef and big $\mathbb{R}$-divisor. Let A be an interior element of D. As in [2, Theorem 3.9.1], there is a birational contraction

$$\sigma : X \to X'$$

such that a curve $\ell \subset X$ is contracted to a point if and only if the class $[\ell]$ is contained in C, and such that $A = \sigma^* A'$ for some ample $\mathbb{R}$-divisor A'.

By the projection formula and since A is contained in D, we have $A'.c_2(X') = \sigma^* A'.c_2(X)$ $= A.c_2(X) = 0$. For any ample $\mathbb{R}$-divisor P on X', a small perturbation $A'_\varepsilon := A' - \varepsilon P$ of the ample divisor A' is still ample because the ample cone of X' is open. By Miyaoka's pseudo-effectivity of c_2 for minimal variety, we have

$$0 \le \varepsilon P.c_2(X') \le (A'_\varepsilon + \varepsilon P).c_2(X') = A'.c_2(X') = 0.$$

So $P.c_2(X') = 0$. Since $\mathrm{NS}_{\mathbb{C}}(X')$ is spanned by ample divisors, we obtain then $c_2(X') = 0$ as a linear form on $\mathrm{NS}_{\mathbb{C}}(X')$.

Thus, $c_1(X)$ and $c_2(X)$ vanish, and by [18, Corollary, p. 266], we have $X' = T/F$ where F is a finite group acting on the abelian variety T freely outside a finite set. Since D and hence C are stable under the action of $B := \mathrm{Aut}(X)$, the contraction

$\sigma : X \to X'$ is B-equivariant. By [1, §3, especially Proposition 3] applied to étale-in-codimension-one covers, replacing T by the finite cover corresponding to the maximal lattice in $\pi_1(X' \setminus \operatorname{Sing} X')$, we can lift the action of B on X' to an action of a group $\widetilde{B}$ on T such that $\widetilde{B}/\operatorname{Gal}(T/X') \cong B$. This proves Theorem 4.1 under condition (1).

Next, assume condition (2) in Theorem 4.1. The maximality of the rank of $G/N(G)$ and [23, Lemma 2.11] imply the Kodaira dimension $\kappa(X) = 0$. The abundance theorem for minimal threefolds implies $K_X \sim_{\mathbb{Q}} 0$ (cf. [12, 3.13]). Replacing G by a finite-index subgroup, we may assume that $G \mid \operatorname{NS}_{\mathbb{C}}(X)$ is solvable and has connected Zariski-closure in $\operatorname{GL}(\operatorname{NS}_{\mathbb{C}}(X))$ (cf. Theorem 2.2 or 2.3). By [26, Claim 2.5(1)], $c_2(X)$ is perpendicular to a nef and big $\mathbb{R}$-divisor. We are reduced to condition (1). This proves Theorem 4.1. □

The next is the key step towards Theorem 1.1.

We recall the notation in the Introduction: For a subgroup G of $\operatorname{Aut}(X)$, let $\overline{G} \subseteq \operatorname{GL}(\operatorname{NS}_{\mathbb{C}}(X))$ be the Zariski-closure of $G \mid \operatorname{NS}_{\mathbb{C}}(X)$ and $R(\overline{G})$ its solvable radical, both of which are defined over $\mathbb{Q}$. We have a natural composition of homomorphisms:

$$\iota : G \to G \mid \operatorname{NS}_{\mathbb{C}}(X) \to \overline{G}.$$

Theorem 4.2. *Let X be a 3-dimensional minimal projective variety and $G \le \operatorname{Aut}(X)$ a subgroup such that $G \mid \operatorname{NS}_{\mathbb{C}}(X)$ is not virtually solvable. Then we have:*

(1) *Suppose that $R(G) := \iota^{-1}(\iota(G) \cap R(\overline{G}))$ is of null entropy. Then $R(G) \mid \operatorname{NS}_{\mathbb{C}}(X)$ is virtually unipotent and hence of null entropy. Replacing G by a suitable finite-index subgroup, $G/R(G)$ is embedded as a Zariski-dense subgroup in $H := \overline{G}/R(\overline{G})$ so that $H(\mathbb{R})$ is a semi-simple real linear algebraic group and is either of real rank* 1 (cf. [14, 0.25]) *or locally isomorphic to* $\operatorname{SL}_3(\mathbb{R})$ *or* $\operatorname{SL}_3(\mathbb{C})$.

(2) *Suppose that $R(G)$ is not of null entropy. Set $B := \operatorname{Aut}(X)$. Then there is a B-(and hence G-) equivariant birational surjective morphism $X \to X'$ such that $X' \cong T/F$ for a finite group F acting freely outside a finite set of an abelian variety T of dimension three. Further, the action of B on X' lifts to an action of a group $\widetilde{B}$ on T such that $\widetilde{B}/F \cong B$.*

We now prove Theorem 4.2. Note that $\iota(G)$ is contained in $\overline{G}(\mathbb{Q})$, and is Zariski-dense in $\overline{G}$. As in Lemma 2.5, replacing G by a suitable finite-index subgroup, we may assume $\overline{G}$ is connected. Set $R := R(G)$. The $\iota : G \to \overline{G}$ above induces an injective homomorphism:

$$\gamma : G/R \to H := \overline{G}/R(\overline{G}).$$

Indeed, γ is defined over $\mathbb{Q}$ (cf. [14, 0.11]). Of course, H is semi-simple. $R \mid \operatorname{NS}_{\mathbb{C}}(X)$ is solvable, being embedded in the solvable group $R(\overline{G})$.

Lemma 4.3. *Up to finite index, $H(\mathbb{R})$ is either semi-simple and of real rank one, or locally isomorphic to $\mathrm{SL}_3(\mathbb{R})$ or $\mathrm{SL}_3(\mathbb{C})$.*

Proof. Let S be a Levi subgroup of $\overline{G}$ such that $\overline{G} = R(\overline{G})S$. Our S can be chosen to be defined over $\mathbb{Q}$ and the induced composite homomorphism $S \to \overline{G} \to H$ is a $\mathbb{Q}$-isogeny (cf. [4, Proof of Proposition 11.23]). Now the argument in [7, Theorem 5.1, Proposition 5.2] for the action $S(\mathbb{R}) \mid \mathrm{NS}_{\mathbb{R}}(X)$ ($\mathbb{Q}$-isogeny to $H(\mathbb{R})$) implies that either $H(\mathbb{R})$ is of real rank ≤ 1, or $H(\mathbb{R})$ is of real rank ≥ 2 and is locally isomorphic to $\mathrm{SL}_3(\mathbb{R})$ or $\mathrm{SL}_3(\mathbb{C})$. In fact, our $S(\mathbb{R})$ and indeed even the larger group $\overline{G}(\mathbb{R})$ already act on $\mathrm{NS}_{\mathbb{R}}(X)$ as the extension of the geometrically induced action of the Zariski-dense subgroup $G \mid \mathrm{NS}_{\mathbb{R}}(X)$ of $\overline{G}(\mathbb{R})$, so we do not need Margulis' condition on $G \mid \mathrm{NS}_{\mathbb{R}}(X)$ there, for the extension of the action. To be precise, one main purpose of the extra assumption in [7] on the rank of a lattice (acting on X) of a semi-simple real Lie group is to extend the action of the lattice on cohomology groups of X to an action of the real Lie group.

If H has real rank $\mathrm{rk}_{\mathbb{R}} H = 0$, then $H(\mathbb{R})$ is compact. The image of G in $H(\mathbb{R})$ is contained in an arithmetic subgroup of H, hence discrete and finite. This image is Zariski-dense in $H(\mathbb{R})$. Thus $H(\mathbb{R})$ is a finite group and hence G is a finite extension of R, so $G \mid \mathrm{NS}_{\mathbb{C}}(X)$ is virtually solvable, which contradicts the assumption. Thus, $\mathrm{rk}_{\mathbb{R}} H$ is at least one and the lemma is proved. □

We return to the proof of Theorem 4.2. Suppose that R is of null entropy. As mentioned in the proof of Theorem 2.2 (cf. [16, Proposition 2.2]) the set

$$U(R) := \{g \in R\,;\, g^* \mid \mathrm{NS}_{\mathbb{C}}(X) \text{ is unipotent}\}$$

is a finite-index subgroup of R. So Theorem 4.2(1) occurs by the lemma above.

Hence we may assume that R is not of null entropy. We will deduce Theorem 4.2(2). Take $R_1 \le R$ a finite-index subgroup such that $R_1 \mid \mathrm{NS}_{\mathbb{C}}(X)$ has connected Zariski-closure in $\mathrm{GL}(\mathrm{NS}_{\mathbb{C}}(X))$ (and is solvable). Thus $R_1/N(R_1) \cong \mathbb{Z}^{\oplus r}$ for some $1 \le r \le \dim X - 1 = 2$ by [23, Theorem 1.2]. If $r = 2$, then Theorem 4.2(2) holds by Theorem 4.1.

Thus we may assume $r = 1$, i.e., $\langle \overline{h} \rangle = R_1/N(R_1) \cong \mathbb{Z}$ with $h \in R_1$ of positive entropy.

Lemma 4.4. $K_X \sim_{\mathbb{Q}} 0$.

Proof. Suppose the contrary that the lemma is false. By the 3-dimensional minimal model program and abundance theorem (cf. [12, 3.13]), the Kodaira dimension $\kappa(X)$ is positive, and $|mK_X|$ (for some $m > 0$) defines a holomorphic map $\psi : X \to Y$ with connected fibres and $\dim Y = \kappa(X)$. The induced action of G on Y is trivial if G is replaced by a suitable finite-index subgroup (cf. [20, Theorem 14.10]). Hence G acts faithfully on a general fibre S of ψ. Under the identification $G \cong G \mid S$, we have $N(G \mid S) = N(G) \mid S$

(cf. [24, 2.1(11) Remark]). Since $G \neq N(G)$, our $G \mid S$ is not of null entropy. Hence $\dim S \geq 2$. Also $\dim S = \dim X - \dim Y \leq \dim X - 1 = 2$. Thus $\dim S = 2$.

In the notation above, $(R_1 \mid S)/N(R_1 \mid S) \cong \mathbb{Z}$. Hence the restrictions of R_1 and R on $\mathrm{NS}_{\mathbb{C}}(S)$ are virtually solvable (cf. Theorem 2.2 or 2.3). Lemma 2.8 is applicable to $R \mid S \lhd G \mid S$ since the conditions in Lemma 2.8 (2) (for surfaces) follow from the condition in Lemma 2.8 (1). Indeed, by the cone theorem of Lie-Kolchin type [23, Theorem 1.6], R_1 (replaced by a finite-index subgroup) has a common nonzero nef eigenvector L_1; thus the class h^*L_1 is parallel to L_1; after switching h with h^{-1} if necessary, [22, Lemma 2.12] implies that $h^*L_1 = d_1(h)L_1$, and also the second condition in Lemma 2.8 (2). So, by Lemma 2.8, replacing G by its subgroup of index ≤ 2, L_h gives rise to a character $\chi : G \mid S \to (\mathbb{R}_{>0}, \times)$ and that (for surfaces) $\mathrm{Ker}(\chi) = N(G \mid S)$. So the null set $N(G \mid S)$ is a subgroup and $G/N(G) = (G \mid S)/N(G \mid S) \cong \mathrm{Im}\,\chi$, an abelian group. Hence $G \mid L_{\mathbb{C}}$ is virtually solvable (cf. Theorem 2.2 or 2.3), contradicting the assumption. □

In notation of 2.1, there exist two nonzero nef divisors L_h, $L_{h^{-1}}$ (which will be fixed) such that $(h^{\pm 1})^* L_{h^{\pm 1}} = d_1(h^{\pm 1}) L_{h^{\pm 1}}$ with $d_1(h^{\pm 1}) > 1$.

Claim 4.5. *Suppose there are a nef $\mathbb{R}$-divisor M, a real number $\lambda \neq 1$ and an integer $s \neq 0$ such that $(h^s)^*M = \lambda M$ and M is not parallel to $L_{h^{\pm 1}}$. Then Theorem 4.2(2) holds.*

Proof. Note that $(h^s)^*L_h = d_1(h)^s L_h$, $(h^s)^*L_{h^{-1}} = d_1(h^{-1})^{-s} L_{h^{-1}}$, and $(h^s)^*M = \lambda^s M$. Rewriting h^s as h, we may assume $s = 1$. We have $M.c_2(X) = h^*(M.c_2(X)) = h^*M.h^*c_2(X) = \lambda M.c_2(X)$. Hence $M.c_2(X) = 0$ for $\lambda \neq 1$. Similarly, $L_{h^{\pm 1}}.c_2(X) = 0$. By the assumption, $M.L_{h^{\pm 1}} \neq 0$ (cf. [8, Corollary 3.2]). Thus, since $M, L_h, L_{h^{-1}}$ are nef eigenvectors of h^* corresponding to eigenvalues λ, $d_1(h)$, $1/d_1(h^{-1})$ and since $d_1(h) \neq 1/d_1(h^{-1})$, [8, Lemma 4.4] implies that the product of these three nef divisors is nonzero and hence the sum of these three is a nef and big divisor, perpendicular to $c_2(X)$. So Theorem 4.2(2) holds true, by Theorem 4.1(1) and Lemma 4.4. □

We return to the proof of Theorem 4.2. As proved above, the closed cone $\mathrm{Nef}(X) \cap c_2(X)^\perp = \{M \in \mathrm{Nef}(X) \mid M.c_2(X) = 0\}$ contains $L_{h^{\pm 1}}$. Since $R_1 \mid \mathrm{NS}_{\mathbb{C}}(X)$ is solvable, the cone theorem of Lie-Kolchin type (cf. e.g. [23, Theorem 2.6]) implies that the above closed cone contains a nonzero common nef divisor L_1 (with $L_1.c_2(X) = 0$) of R_1, after R_1 is replaced by a finite-index subgroup. Write $g^*L_1 = \chi(g)L_1$ and consider the homomorphism

$$\varphi : R_1 \to (\mathbb{R}, +), \quad g \mapsto \log \chi(g).$$

Clearly, $N(R_1) \leq \mathrm{Ker}(\varphi)$ (cf. 2.1). If $g \in \mathrm{Ker}(\varphi) \setminus N(R_1)$, then the product of the three nef eigenvectors L_1, $L_{g^{\pm 1}}$ (corresponding to different eigenvalues 1, $d_1(g) \neq 1/d_1(g^{-1})$ of g^*) is nonzero by [8, Lemma 4.4] and hence the sum of these three vectors is a nef and

big $\mathbb{R}$-divisor class perpendicular to $c_2(X)$. Thus, by Lemma 4.4, we can apply Theorem 4.1(1) to conclude Theorem 4.2(2).

Therefore, we may assume that $\mathrm{Ker}(\varphi) = N(R_1)$. In particular, $\chi(h) \neq 1$, where $h^*L_1 = \chi(h)L_1$. Possibly switching h with h^{-1}, we may assume that $\chi(h) > 1$. By Claim 4.5, we may assume that $L_1 = L_h$ which is a common eigenvector of R_1. Thus the condition (1) of Lemma 2.8 is satisfied while the condition (2) can be assumed in view of Claim 4.5. So, by Lemma 2.8, replacing G by its subgroup of index ≤ 2, we may assume that G fixes L_h up to scalars. Write $g^*L_h = \chi'(g)L_h$. Let

$$\psi : G \to (\mathbb{R}, +), \quad g \mapsto \log \chi'(g)$$

so that $G/\mathrm{Ker}(\psi)$ is mapped to an abelian subgroup of $(\mathbb{R}, +)$. If $\mathrm{Ker}(\psi) \neq N(G)$, then as in the case of $\mathrm{Ker}(\varphi) \neq N(G)$ above, we take $g \in \mathrm{Ker}(\psi) \setminus N(G)$, so $c_2(X)$ is perpendicular to the nef and big divisor $L_1 + L_g + L_{g^{-1}}$ and hence Theorem 4.2(2) occurs.

Thus we may assume that $N(G) = \mathrm{Ker}(\psi)$ which is hence a subgroup of G. Since $G/N(G) \cong \mathrm{Im}\,\psi$ is abelian, $G \,|\, \mathrm{NS}_{\mathbb{C}}(X)$ is virtually solvable by Theorem 2.2 or 2.3, contradicting the assumption. *The proof of Theorem* 4.2 *is completed.*

4.6. Proof of Theorem 1.1

We may assume that Theorem 4.2(2) occurs and use the notation there. Let $\widetilde{G}$ be the lifting to T of $G \,|\, X'$ with $\widetilde{G}/F = G \,|\, X'$. As sets (and set of left cosets), we have equalities $N(\widetilde{G})/F = N(G \,|\, X') = N(G) \,|\, X'$; so $N(G) \le G$ if and only if $N(\widetilde{G}) \le \widetilde{G}$, and if this is the case $\widetilde{G}/N(\widetilde{G}) \cong G/N(G)$ (cf. [24, Lemma 2.6]). Thus, by Theorem 2.3(3), as on X, neither $G \,|\, \mathrm{NS}_{\mathbb{C}}(X')$ nor $\widetilde{G} \,|\, \mathrm{NS}_{\mathbb{C}}(T)$ is virtually solvable. By the same reasoning, the lifting to T of $R(G) \,|\, X'$ has virtually solvable action on $\mathrm{NS}_{\mathbb{C}}(T)$, is normal in $\widetilde{G}$ and is not of null entropy. $R(\widetilde{G})$ contains this lifting up to finite index, so it is not of null entropy. Hence we may assume that $X = T$, a complex torus.

Let $\hat{G} \le \mathrm{GL}(H^0(X, \Omega_X^1)^\vee) = \mathrm{GL}_3(\mathbb{C})$ be the Zariski-closure of the action $G \,|\, H^0(X, \Omega_X^1)^\vee$. Since every $g \in G$ acts on $H^1(X, \mathbb{Z})$ invertibly, its matrix representation has determinant ± 1; note also $H^1(X, \mathbb{C}) = H^0(X, \Omega_X^1) \oplus H^0(X, \Omega_X^1)^\vee$; hence we may assume that $\hat{G}$ is contained in $\mathrm{SL}_3(\mathbb{C})$ and connected, after G is replaced by a finite-index subgroup.

Since $H^*(X, \mathbb{C}) := \oplus_{i \ge 0} H^i(X, \mathbb{C})$ is generated by wedge products of $H^0(X, \Omega_X^1)$ and its conjugate, the null set $N(G)$ is equal to $\{g \in G\,;\, g \,|\, H^0(X, \Omega_X^1) \text{ is of null entropy}\}$. Let

$$R := G \cap R(\hat{G}) \lhd G, \quad U := G \cap U(\hat{G}) \lhd G.$$

Then $R(\hat{G}) \,|\, H^*(X, \mathbb{C})$ and hence $R \,|\, H^*(X, \mathbb{C})$ are solvable. By Theorem 2.3, R has a finite-index subgroup R_1 such that

$$\mathbb{Z}^{\oplus r} \cong R_1/N(R_1) \le R/N(R).$$

Also $|N(R) : U| < \infty$ (cf. Theorem 2.2). Thus R/U contains a copy of $\mathbb{Z}^{\oplus r}$ as a subgroup of finite index (cf. Lemma 2.4). Consider the natural embedding

$$G/U \to J := \hat{G}/U(\hat{G})$$

into the reductive group J of real rank $\le \mathrm{rk}_{\mathbb{R}} \mathrm{SL}_3(\mathbb{C}) = 2$.

If $r \ge 2$, then $\mathrm{rk}_{\mathbb{R}} J = 2$; the Zariski-closure of $R/U \subset J$ contains a copy of $\mathbb{Z}^{\oplus 2}$ and hence a maximal torus of J, and is normal in J, so this closure equals J. Thus J (like R/U) and hence the actions of G/U and G on $H^*(X, \mathbb{C})$ are all solvable, contradicting the assumption.

Consider the case $r = 0$, i.e., $R \subseteq N(G)$. This contradicts the extra assumption that $R(G)$ is not of null entropy and the assertion(*): $R(G)$ equals R up to finite index. Indeed, as in the proof of Lemma 2.6, $G \,|\, \mathrm{NS}_{\mathbb{C}}(X) = G/K$ with $|K : G \cap \mathrm{Aut}_0(X)| < \infty$, while $G \,|\, H^0(X, \Omega_X^1)^\vee = G/(G \cap \mathrm{Aut}_0(X))$. Hence $G \,|\, \mathrm{NS}_{\mathbb{C}}(X)$ equals $G \,|\, H^0(X, \Omega_X^1)^\vee$ modulo a finite group. Now the assertion(*) follows from the definitions of $R(G)$ and R.

Finally, assume $r = 1$, i.e., $R_1/N(R_1) \cong \mathbb{Z}$. For any $g_1 \in G$, the group $G_1 := \langle g_1, R\rangle$ (replaced by its finite-index subgroup) has $G_1 \,|\, H^*(X, \mathbb{C})$ solvable and $G_1/N(G_1) \cong \mathbb{Z}^{\oplus s}$ (cf. Theorem 2.3). We claim that $s \ge 2$ for some g_1. If the claim is false, then for any $g_1 \in G$, we have $s \le 1$ and hence $g_1^a = h^b n$ for some $a \ge 1$, $n \in N(G)$, where $\langle \bar{h}\rangle = R_1/N(R_1) \le G_1/N(G_1)$. Thus g_1 (mod R) has a positive power acting as a unipotent element on $H^0(X, \Omega_X^1)^\vee$ because the same is true for $n \in N(G)$. So the subgroup G/R of an arithmetic subgroup of $\hat{G}/R(\hat{G})$ (defined over $\mathbb{Q}$; cf. [14, ChI; 0.11]) has a unipotent group $U(G/R)$ as its subgroup of finite index, by Burnside's theorem as in [16, Proposition 2.2] or Theorem 2.2. Thus its Zariski-closure $\hat{G}/R(\hat{G})$ is both unipotent and semi-simple and hence trivial. So $G \,|\, H^*(X, \mathbb{C})$ is solvable, contradicting the assumption.

Thus the claim is true and hence some $G_1 := \langle g_1, R\rangle$ has $G_1/N(G_1) \cong \mathbb{Z}^{\oplus s}$ for some $s \ge 2$. So by [26, Paragraph before §2.8], $U(G_1)$ and hence $N(G_1)$ and $N(R)$ act as finite groups on $H^0(X, \Omega_X^1)$ and also on $H^*(X, \mathbb{C})$ (cf. Proof of Theorem 2.2). Thus, when restricted on $H^0(X, \Omega_X^1)^\vee$, our R (containing a finite-index subgroup R_1 with $R_1/N(R_1) \cong \mathbb{Z}$) is virtually infinite cyclic and normalized by G, so it is contained in the centre of $G \,|\, H^0(X, \Omega_X^1)^\vee$ and of $\hat{G}$, by replacing G by a finite-index subgroup and considering the conjugate action on the derived series of R.

Now we follow referee's suggestion. Take an element $h \in R \setminus N(R)$. If $h \,|\, H^0(X, \Omega_X^1)^\vee \in \mathrm{SL}_3(\mathbb{C})$ has three distinct eigenvectors, then h and all elements of G are simultaneously diagonalizable and hence $G \,|\, H^*(X, \mathbb{C})$ is abelian, contradicting the assumption.

Therefore, relative to a suitable basis B of $H^0(X, \Omega_X^1)^\vee$, our $h \mid H^0(X, \Omega_X^1)^\vee$ is in one of the Jordan canonical forms

$$\operatorname{block\,diag}[\alpha^{-2}, J_2(\alpha)], \quad \operatorname{diag}[\alpha^{-2}, \alpha, \alpha]$$

and the matrix representation $g \mid H^0(X, \Omega_X^1)^\vee = (a_{ij})$ of every $g \in G$ is especially upper triangular. Consider the projection

$$\tau : G \to \mathbb{C}^*, \quad g \mapsto a_{11}.$$

If $\operatorname{Ker}\tau \subseteq N(G)$, then the actions of $\operatorname{Ker}\tau$ and hence of $(\operatorname{Ker}\tau)\,R$ on $H^0(X, \Omega_X^1)^\vee$ are virtually solvable (cf. Theorem 2.2), so is that of G, because $G/((\operatorname{Ker}\tau)\,R)$ is a quotient of the abelian group $\operatorname{Im}\tau$. This contradicts the assumption.

Thus, we can take $g_1 \in \operatorname{Ker}\tau \setminus N(G)$. Then $g_1 \mid H^0(X, \Omega_X^1)^\vee$ has 3 eigenvalues $1, \lambda^{\pm 1}$ (with $|\lambda| \neq 1$); it has a unique (up to scalar) eigenvector $w \in H_1(X, \mathbb{Z})$ ($=$ the lattice Λ of the torus $X = \mathbb{C}^3/\Lambda$) corresponding to the eigenvalue $1 \in \mathbb{Q}$ and is proportional to the column vector $(1, 0, 0)^t$ (in basis B). Now h or h^{-1} takes w to $\alpha^{-2}w$ with $|\alpha^{-2}| < 1$. This contradicts the fact that $h(\Lambda) = \Lambda$ which is discrete in $\mathbb{C}^3 = \mathbb{R}^6$. *Theorem* 1.1 *is proved.*

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